Eq. de Cauchy

$$\rho \frac{D\underset{\sim}{v}}{Dt} = \nabla \cdot \underset{=}{\sigma} + \rho \underset{\sim}{g} \qquad (1)$$

↳ tensor de tensões ?

Como modelar $\underset{=}{\sigma}$?

↳ eq. constitutiva

Versão Diferencial do princípio de Balanço de momento angular

T.T.R → $\frac{D}{Dt} \int_V G \, dV = \int_V \left(\frac{DG}{Dt} + G \nabla \cdot \underset{\sim}{v} \right) dV$

$$\boxed{G = \underset{\sim}{x} \times \rho \underset{\sim}{v}} \rightsquigarrow T.T.R$$

$$\underbrace{\frac{D}{Dt}\int_V \underset{\sim}{x} \times \rho \underset{\sim}{v} \, dV = \sum \underset{\sim}{T} \quad (2)}_{\text{lei física}}$$

Aplicando o T.T.R no primeiro termo do lado esquerdo:

$$\frac{D}{Dt}\int_V \underset{\sim}{x} \times \rho \underset{\sim}{v} \, dV = \underbrace{\int_V \left[\frac{D}{Dt}(\underset{\sim}{x} \times \rho \underset{\sim}{v}) + (\underset{\sim}{x} \times \rho \underset{\sim}{v}) \nabla \cdot \underset{\sim}{v}\right] dV}$$

↳ muito complicado

Note que: $G = \rho A$

T.T.R: $$\frac{D}{Dt}\int_V \rho A \, dV = \int_V \left[\frac{D}{Dt}(\rho A) + \rho A \nabla \cdot \underset{\sim}{v}\right] dV$$

$$\frac{D}{Dt}(\rho A) + \rho A \nabla \cdot \underset{\sim}{v} = \rho \frac{DA}{Dt} + A \frac{D\rho}{Dt} + \rho A \nabla \cdot \underset{\sim}{v}$$

$$= \underbrace{A \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \underset{\sim}{v} \right)}_{0,\ \text{eq. da continuidade}} + \rho \frac{DA}{Dt}$$

Logo:

$$\boxed{\frac{D}{Dt} \int_V \rho A \, dV = \int_V \rho \frac{DA}{Dt} \, dV}$$

↳ Consequência do T.T.R + eq. da continuidade

Assim:

$$\int_V \rho \frac{D}{Dt} (\underset{\sim}{x} \times \underset{\sim}{v}) \, dV = \sum \underset{\sim}{T} \qquad (3)$$

↓ torques atuando numa partícula fluida

$$\sum \underset{\sim}{T} = \sum \underset{\sim}{T_s} + \sum \underset{\sim}{T_v} \rightarrow \text{volume} \qquad (4)$$

↳ superfície

$$\sum \underset{\sim}{T_s} = \int_S \underset{\sim}{x} \times (\hat{n} \cdot \underline{\underline{\sigma}})\, dS \qquad (5)$$

$$\sum \underset{\sim}{T_v} = \int_V \left(\underset{\sim}{x} \times \rho \underset{\sim}{g} + \underset{\sim}{t_v} \right) dV \qquad (6)$$

densidade volumétrica de torques internos (intrínsecos) ao material

Usando (3), (4), (5) e (6):

$$\int_V \rho \underbrace{\frac{D}{Dt}\left(\underset{\sim}{x} \times \underset{\sim}{v} \right)}_{\underset{\sim}{x} \times \frac{D\underset{\sim}{v}}{Dt} + \underset{\sim}{v} \times \underbrace{\frac{D\underset{\sim}{x}}{Dt}}_{\underset{\sim}{v}} = \underset{\sim}{x} \times \frac{D\underset{\sim}{v}}{Dt}} dV = \int_V \left(\underset{\sim}{x} \times \rho \underset{\sim}{g} + \underset{\sim}{t_v} \right) dV + \int_S \underset{\sim}{x} \times (\hat{n} \cdot \underline{\underline{\sigma}})\, dS$$

$$\int_V \underset{\sim}{x} \times \left(\rho \frac{D\underset{\sim}{v}}{Dt} - \rho \underset{\sim}{g} - \underset{\sim}{t_v} \right) dV = \int_S \underset{\sim}{x} \times (\hat{n} \cdot \underline{\underline{\sigma}})\, dS \qquad (7)$$

Note que: $\underset{\sim}{x} \times (\hat{n} \cdot \underline{\underline{\sigma}}) = x_a \hat{e}_a \times (\hat{n}_b \hat{e}_b \cdot \sigma_{cd} \hat{e}_c \hat{e}_d)$

$= x_a \hat{e}_a \times (\hat{n}_b \sigma_{cd} \delta_{bc} \hat{e}_d) = x_a \hat{n}_b \sigma_{bd} \hat{e}_a \times \hat{e}_d$

$= x_a \hat{n}_b \sigma_{bd} \varepsilon_{ade} \hat{e}_e = \varepsilon_{ea} x_a \hat{n}_b \sigma_{bd}$

$\Rightarrow \varepsilon_{ijk} x_j \hat{n}_p \sigma_{pk} = \underbrace{\varepsilon_{ijk} x_j \sigma_{pk}}_{T_{ip}} \hat{n}_p$

T.D. p/ tensores de 2ª ordem:

$$\int_S T_{ip} \hat{n}_p \, dS = \int_S \hat{n} \cdot \underline{\underline{T}}^T dS \underset{T.D}{=} \int_V \nabla \cdot \underline{\underline{T}}^T dV$$

Logo:

$$\int_S \underset{\sim}{x} \times (\hat{n} \cdot \underline{\underline{\sigma}}) dS = \int_V \frac{\partial}{\partial x_p} \left(\varepsilon_{ijk} x_j \sigma_{pk} \right) dV$$

$$= \int_V \varepsilon_{ijk} \left(\underbrace{\frac{\partial x_j}{\partial x_p}}_{\delta_{jp}} \sigma_{pk} + x_j \frac{\partial}{\partial x_p} \sigma_{pk} \right) dV$$

$$= \int_V \left(\varepsilon_{ijk} \sigma_{jk} + \varepsilon_{ijk} x_j \frac{\partial \sigma_{pk}}{\partial x_p} \right) dV$$

$$\int_S \boldsymbol{x} \times (\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma})\, dS = \int_V \left[\boldsymbol{\varepsilon} : \boldsymbol{\sigma} + \boldsymbol{x} \times (\nabla \cdot \boldsymbol{\sigma}) \right] dV \qquad (8)$$

$(8) \rightarrow (7)$:

$$\int_V \boldsymbol{x} \times \left[\rho \frac{D\boldsymbol{v}}{Dt} - \rho \boldsymbol{g} - \boldsymbol{t}_V \right] dV = \int_V \left[\boldsymbol{\varepsilon} : \boldsymbol{\sigma} + \boldsymbol{x} \times (\nabla \cdot \boldsymbol{\sigma}) \right] dV$$

$$\int_V \left\{ \boldsymbol{x} \times \underbrace{\left[\rho \frac{D\boldsymbol{v}}{Dt} - \rho \boldsymbol{g} - \nabla \cdot \boldsymbol{\sigma} \right]}_{\boldsymbol{0} \,\rightarrow\, \text{eq de Cauchy}} - \boldsymbol{t}_V - \boldsymbol{\varepsilon} : \boldsymbol{\sigma} \right\} dV = \boldsymbol{0}$$

$$\boxed{\boldsymbol{\varepsilon} : \boldsymbol{\sigma} = -\boldsymbol{t}_V} \rightarrow \text{o que isto significa?}$$

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma} = \varepsilon_{ijk}\, \sigma_{jk}$$

$$\begin{tikzcd} & 1 \arrow[dr, bend left] & \\ 3 \arrow[ur, bend left] & & 2 \arrow[ll, bend left] \end{tikzcd}$$

$$\underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} = \left(\overset{+}{\cancel{\varepsilon_{123}}}\,\sigma_{23} + \overset{-}{\cancel{\varepsilon_{132}}}\,\sigma_{32}\right)\hat{e}_1$$

$$+ \left(\overset{+}{\cancel{\varepsilon_{231}}}\,\sigma_{31} + \overset{-}{\cancel{\varepsilon_{213}}}\,\sigma_{13}\right)\hat{e}_2$$

$$+ \left(\overset{+}{\cancel{\varepsilon_{312}}}\,\sigma_{12} + \overset{-}{\cancel{\varepsilon_{321}}}\,\sigma_{21}\right)\hat{e}_3$$

$$= (\sigma_{23} - \sigma_{32})\hat{e}_1 + (\sigma_{31} - \sigma_{13})\hat{e}_2$$

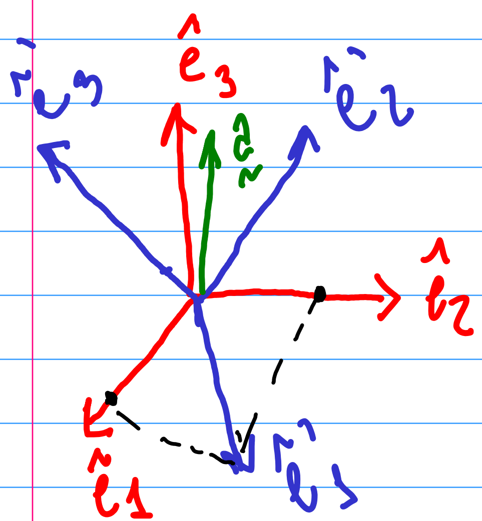
$$+ (\sigma_{12} - \sigma_{21})\hat{e}_3 = -\underset{\sim}{t_v}$$

Se $\underset{\sim}{t_v} = \underset{\sim}{0} \rightarrow$
$$\left.\begin{aligned} \sigma_{23} &= \sigma_{32} \\ \sigma_{31} &= \sigma_{13} \\ \sigma_{12} &= \sigma_{21} \end{aligned}\right\} \boxed{\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T}$$

"Na ausência de forças internas, intrínsecos ao material, o tensor de tensões do mesmo será sempre simétrico." → Importante

Transformações ortogonais

Considere dois conjuntos de eixos ortonormais com a mesma origem:



Sabemos que:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

e

$$\hat{e}_i' \cdot \hat{e}_j' = \delta_{ij}$$

Seja representar $\underset{\sim}{a}$ nos sistemas $\hat{e}_i$ e $\hat{e}_i'$

$$\underset{\sim}{a} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 = a_1' \hat{e}_1' + a_2' \hat{e}_2' + a_3' \hat{e}_3'$$

$\underset{\sim}{a}$ é invariante. O que muda são suas componentes em cada escolha de S.C.

Considere escrever $\hat{e}_i'$ em termos de $\hat{e}_i$:

$$
\begin{aligned}
\hat{e}_1' &= (\hat{e}_1' \cdot \hat{e}_1)\hat{e}_1 + (\hat{e}_1' \cdot \hat{e}_2)\hat{e}_2 + (\hat{e}_1' \cdot \hat{e}_3)\hat{e}_3 \\
\hat{e}_2' &= (\hat{e}_2' \cdot \hat{e}_1)\hat{e}_1 + (\hat{e}_2' \cdot \hat{e}_2)\hat{e}_2 + (\hat{e}_2' \cdot \hat{e}_3)\hat{e}_3 \\
\hat{e}_3' &= (\hat{e}_3' \cdot \hat{e}_1)\hat{e}_1 + (\hat{e}_3' \cdot \hat{e}_2)\hat{e}_2 + (\hat{e}_3' \cdot \hat{e}_3)\hat{e}_3
\end{aligned}
\qquad (9)
$$

$$\hookrightarrow \hat{e}_i' = \underbrace{(\hat{e}_i' \cdot \hat{e}_j)}_{\Phi_{ij}} \hat{e}_j \quad \Big\} \quad \hat{e}_i' = \Phi_{ij}\hat{e}_j$$

$\Phi_{ij} \downarrow$ tensor ortogonal

Transformação de um vetor entre sistemas de coordenadas

Note que: $\underset{\sim}{a} = a_1'\hat{e}_1' + a_2'\hat{e}_2' + a_3'\hat{e}_3' \qquad (10)$

escrevendo (10) em termos de $\hat{e}_i'$ e usando (9):

$$\underset{\sim}{a} = \left[a_1'(\hat{e}_1'\cdot\hat{e}_1) + a_2'(\hat{e}_2'\cdot\hat{e}_1) + a_3'(\hat{e}_3'\cdot\hat{e}_1)\right]\hat{e}_1$$

$$\underbrace{+ \left[a_1'(\hat{e}_1'\cdot\hat{e}_2) + a_2'(\hat{e}_2'\cdot\hat{e}_2) + a_3'(\hat{e}_3'\cdot\hat{e}_2)\right]\hat{e}_2 + \left[a_1'(\hat{e}_1'\cdot\hat{e}_3) + a_2'(\hat{e}_2'\cdot\hat{e}_3) + a_3'(\hat{e}_3'\cdot\hat{e}_3)\right]\hat{e}_3}$$

$$a_i = a_j'\underbrace{(\hat{e}_j'\cdot\hat{e}_i)}_{\varphi_{ji}} \Rightarrow \boxed{\begin{array}{c}\underbrace{a_i = a_j'\,\varphi_{ji}} \\ \underset{\sim}{a} = \underset{=}{\varphi}^T\cdot\underset{\sim}{a}'\end{array}}$$

Seja agora escrever $\hat{e}_i$ em termos de $\hat{e}_i'$:

$$\hat{e}_1 = (\hat{e}_1\cdot\hat{e}_1')\hat{e}_1' + (\hat{e}_1\cdot\hat{e}_2')\hat{e}_2' + (\hat{e}_1\cdot\hat{e}_3')\hat{e}_3'$$

$$\hat{e}_2 = (\hat{e}_2\cdot\hat{e}_1')\hat{e}_1' + (\hat{e}_2\cdot\hat{e}_2')\hat{e}_2' + (\hat{e}_2\cdot\hat{e}_3')\hat{e}_3'$$

$$\hat{e}_3 = (\hat{e}_3\cdot\hat{e}_1')\hat{e}_1' + (\hat{e}_3\cdot\hat{e}_2')\hat{e}_2' + (\hat{e}_3\cdot\hat{e}_3')\hat{e}_3'$$

Como $\underset{\sim}{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3 = a_1'\hat{e}_1' + a_2'\hat{e}_2' + a_3'\hat{e}_3'$

$= \underset{\sim}{a}'$

$$\underset{\sim}{a}' = \left[a_1 (\hat{e}_1 \cdot \hat{e}_1') + a_2 (\hat{e}_2 \cdot \hat{e}_1') + a_3 (\hat{e}_3 \cdot \hat{e}_1') \right] \hat{e}_1'$$
$$+ \left[a_1 (\hat{e}_1 \cdot \hat{e}_2') + a_2 (\hat{e}_2 \cdot \hat{e}_2') + a_3 (\hat{e}_3 \cdot \hat{e}_2') \right] \hat{e}_2'$$
$$\underbrace{+ \left[a_1 (\hat{e}_1 \cdot \hat{e}_3') + a_2 (\hat{e}_2 \cdot \hat{e}_3') + a_3 (\hat{e}_3 \cdot \hat{e}_3') \right] \hat{e}_3'}_{a_i' = a_j (\hat{e}_j \cdot \hat{e}_i') = Q_{ij} a_j}$$

Até agora temos:

Lei de transformação de vetores

$$\boxed{\begin{aligned} \underset{\sim}{a} &= \underset{=}{Q}^T \cdot \underset{\sim}{a}' \\ \underset{\sim}{a}' &= \underset{=}{Q} \cdot \underset{\sim}{a} \end{aligned}} \quad \underset{\sim}{a} = \underset{=}{Q}^T \cdot \underset{=}{Q} \cdot \underset{\sim}{a} = \underset{=}{I} \cdot \underset{\sim}{a}$$

$$\underset{=}{Q}^T = \underset{=}{Q}^{-1}$$ ⇝ Relação de Ortogonalidade

Lei de Transformação p/ Tensores Cartesianos de 2ª ordem

Sejam $\underset{\sim}{a}$ e $\underset{\sim}{b}$ tensores cartesianos de ordem 1,

logo: $a_i' = Q_{ip} a_p$ e $b_j' = Q_{jk} b_k$

e $\quad a_i = Q_{pi}\, a_p' \quad$ e $\quad b_j = Q_{kj}\, b_k'$

Seja calcular $a_i b_j = T_{ij} = Q_{pi}\, Q_{kj} \underbrace{a_p' b_k'}_{T_{pk}'}$

ou

$a_i' b_j' = T_{ij}' = Q_{ip}\, Q_{jk} \underbrace{a_p b_k}_{T_{pk}}$

$$\left.\begin{array}{l} T_{ij}' = Q_{ip}\, Q_{jk}\, T_{pk} \\ \text{ou} \\ T_{ij} = Q_{pi}\, Q_{kj}\, T_{pk}' \end{array}\right\} \quad \boxed{\begin{array}{c} \underline{\underline{T}}' = \underline{\underline{Q}} \cdot \underline{\underline{T}} \cdot \underline{\underline{Q}}^T \\ \text{ou} \\ \underline{\underline{T}} = \underline{\underline{Q}}^T \cdot \underline{\underline{T}}' \cdot \underline{\underline{Q}} \end{array}}$$

$\downarrow$

Lei de transformação p/ tensores cartesianos de 2ª ordem

No caso mais geral p/ tensores de ordem "n":

$$T'_{ijk\ldots p} = Q_{li} Q_{mj} Q_{ok} Q_{Rp} \ldots T_{lmo\ldots R}$$

Exemplo: tensor de 4ª ordem:

$$A'_{ijkl} = Q_{ai} Q_{bj} Q_{ck} Q_{dl} A_{abcd}$$

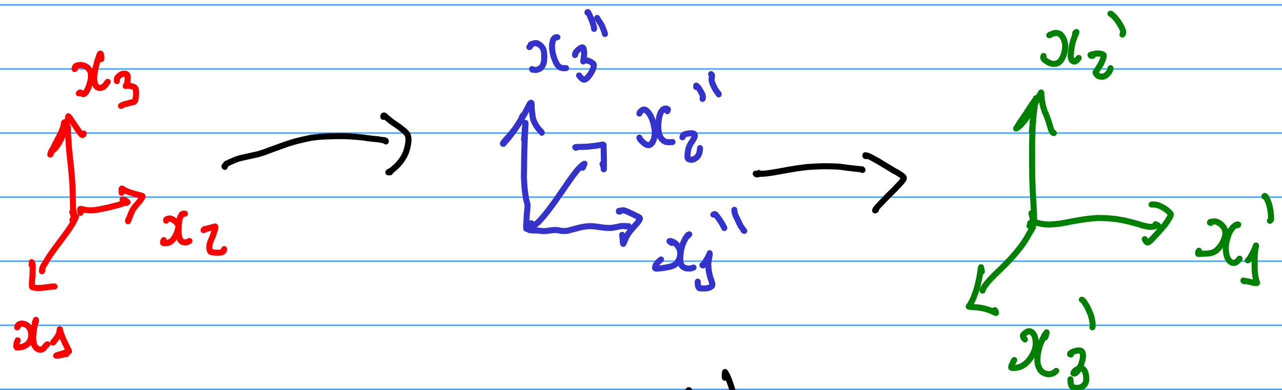
Tensores Isotrópicos

Def: Um tensor cujo os componentes permanecem invariantes a qualquer transformação ortogonal é dito isotrópico.

Teorema: Se T_{ij} são os componentes de um tensor isotrópico de 2ª ordem, então $\boxed{T_{ij} = \alpha \delta_{ij},}$ em que α é um campo escalar.

Demonstração

- Considere $\boxed{T_{ij} = \alpha \delta_{ij}} \longrightarrow T_{pq} = \alpha \delta_{pq}$
- Considere ainda a seguinte rotação de eixos:



$$\hat{e}_i \longrightarrow \hat{e}_i'$$

$$\underbrace{\hat{e}_1' = \hat{e}_2; \quad \hat{e}_2' = \hat{e}_3; \quad \hat{e}_3' = \hat{e}_1}$$

$Q_{12} = Q_{23} = Q_{31} = 1$ e os outros termos são nulos!

$$\underline{\underline{Q}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Se T_{ij} é um tensor cartesiano de 2ª ordem, então o mesmo obedece à seguinte lei de transformação: $T_{ij}' = Q_{ip} Q_{jq} T_{pq}$

Assim: $T'_{ij} = Q_{ip} Q_{jq} \alpha \delta_{pq} = \alpha Q_{ip} Q_{jp}$

Verificando componentes:

$$T'_{11} = \alpha Q_{1p} Q_{1p} = \alpha \left(Q_{11}^2 + Q_{12}^2 + Q_{13}^2 \right) = \alpha$$

$$T'_{22} = \alpha Q_{2p} Q_{2p} = \alpha \left(Q_{21}^2 + Q_{22}^2 + Q_{23}^2 \right) = \alpha$$

$$T'_{33} = \alpha Q_{3p} Q_{3p} = \alpha \left(Q_{31}^2 + Q_{32}^2 + Q_{33}^2 \right) = \alpha$$

$$T'_{12} = \alpha Q_{1p} Q_{2p} = \alpha \left(\underbrace{\cancel{Q_{11} Q_{21}}}_{0} + \underbrace{\cancel{Q_{12} Q_{22}}}_{0} + \underbrace{\cancel{Q_{13} Q_{23}}}_{0} \right)$$

$\vdots$

$$\underbrace{T'_{13} = T'_{21} = T'_{23} = T'_{31} = T'_{32} = T'_{12} = 0}$$

$$\boxed{\underline{\underline{T}}' = \alpha \underline{\underline{I}} = \underline{\underline{T}}}$$

Teorema de Cauchy-Weil

A forma mais geral de representação de um tensor isotrópico de 4ª ordem é:

$$\boxed{\underbrace{T_{ijkl}}_{\text{isotrópico}} = \alpha\, \delta_{ij}\delta_{kl} + \beta\, \delta_{ik}\delta_{jl} + \gamma\, \delta_{il}\delta_{jk}}$$